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TEACHING MATERIAL ON

MATHEMATICS

SCHOOL OF SCIENCE

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Classial Optimization Methods

Classical Optimization Methods

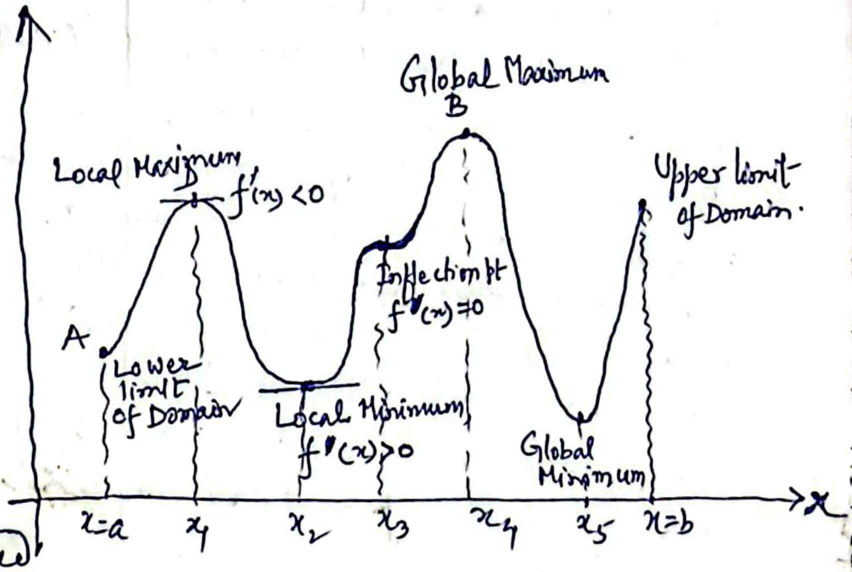
The classical optimization methods are used to obtain an optimal solution of certain types of problems involving continuous and differentiable functions. These methods are analytical in nature and make use of differential calculus to find points of maxima and minima for both unconstrained and constrained continuous objective functions.

In this chapter we shall discuss the necessary and sufficient conditions for obtaining an optimal solution of (i) Unconstrained single & multiple variable optimization problem & (ii) Constrained multivariable optimization problem with equality and inequality constraints.

1. Unconstrained optimization:—

(a) Optimizing single-variable function:—

Let us consider a graph of a continuous function $y = f(x)$ of single independent variable x in the domain (a, b). The domain is the range of values of x. The domain limits (or end points) are generally called



"stationary" (or critical) points. There are two types of stationary points: (a) inflection points and (b) extreme points. The extreme points may be further classified as either local (or relative) or global (or absolute) extrema (maxima or minima).

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Local extreme points represents the maximum or minimum values of the function in the given range of values of the variable. Thus the points $x = a, x_1, x_2, x_3, x_4, x_5$ & b are all extrema of $f(x)$. The classical approach to the Theory of maxima or minima does not provide a direct method of obtaining global (or absolute) maximum or minimum value of a function. It provides only the method for determining the local (or relative) maximum or minimum values.

Soln. Mathematically, a function $y = f(x)$ is said to achieve its maximum value at a point $x = x_0$ if

$f(x_0 + h) - f(x_0) < 0$ or $f(x_0) > f(x_0 + h)$ where h is a sufficiently small no. in the neighbourhood of the point $x = x_0$. In other words, the point x_0 is a local maximum if value of $f(x)$ at every point in the nbd of x_0 does not exceed $f(x_0)$.

Similarly, a function $f(x)$ is said to achieve its minimum value at a point, $x = x_0$ if $f(x_0 + h) - f(x_0) > 0$ or $f(x_0) < f(x_0 + h)$.

If a fn has several local maximum and minimum values then the global minimum (in case of cost minimization) or global maximum (in case of profit maximization) is obtained by comparing the values of the function at various extreme points (including at the limits of the domain). The global minimum value of the function is the minimum value among all local minimum values of the function in the domain. Similarly, the global maximum value of the fn is the maximum value among all local maximum values of the function in the domain. In the fig., the point E, i.e. $f(x_4)$ represents the global maximum, whereas the point F, i.e. $f(x_5)$ represents the global minimum.

The global maxm or minm of a function over the larger interval can also occur at an end point of the interval, rather than at any local (relative) maxm or minm point. It is also possible for a local maximum value of a fn to be less than a local minimum value of the fn.

(§) Conditions for local Minimum and Maximum value

Theorem:- (Necessary Condition) A necessary condition for a point x_0 to be the local extrema (local maximum and minimum) of a function $y = f(x)$ defined in the interval $a \le x \le b$ is that the first derivative of $f(x)$ exists as a finite number at $x = x_0$ and $f'(x_0) = 0$.

Proof:- Let $y = f(x)$ be a given fn which can be expanded in the nhd of $x = x_0$ by Taylor's theorem. Let at $x = x_0$ the value of $f(x)$ be $f(x_0)$.

Consider two values of x, namely $+h$ and $-h$ in the nhd and either side of $x = x_0$ (h being very small). If there is maximum at $x = x_0$ then from definition. $f(x_0) > f(x_0+h)$ and $f(x_0) > f(x_0-h)$. That is $f(x_0+h) - f(x_0)$ and $f(x_0-h) - f(x_0)$ are both negative for maximum at $x = x_0$. Further, if there is minimum at $x = x_0$ then $f(x_0) < f(x_0+h)$ and $f(x_0) < f(x_0-h)$. That is $f(x_0+h) - f(x_0)$ and $f(x_0-h) - f(x_0)$ are both positive for minimum at $x = x_0$.

So by Taylor's theorem we have,

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{\lfloor 2}f''(x_0) + \cdots + \frac{h^n}{\lfloor n}f^n(x_0) + R_n(x_0+\theta h), \quad 0<\theta<1$$

$$\text{or} \quad f(x_0+h) - f(x_0) = hf'(x_0) + \frac{h^2}{\lfloor 2}f''(x_0) + \cdots \quad \text{--- (1)}$$

$$\text{where, } R_n(x_0+\theta h) = \frac{h^{n+1}}{\lfloor n+1} f^{n+1}(x_0+\theta h), \quad 0<\theta<1$$

and is called the remainder.

The expression $f'(x_0)$ and $f''(x_0)$ represents the first and second derivative of $f(x)$ at $x = x_0$.

Similarly $f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{\lfloor 2}f''(x_0) - \cdots$

$$f(x_0-h) - f(x_0) = -hf'(x_0) + \frac{h^2}{\lfloor 2}f''(x_0) + \cdots \quad \text{--- (2)}$$

If h is very small then neglecting the higher order terms of h we get $f(x_0+h) - f(x_0) = hf'(x_0)$ --- (3) & $f(x_0-h) - f(x_0) = -hf'(x_0)$ --- (4)

For $x = x_0$ to be a local maximum or minimum value the sign of $f(x_0+h) - f(x_0)$ and $f(x_0-h) - f(x_0)$ must be the same for all $x = x_0 \pm h$. Thus from eqn ③ and ④ if $f(x_0+h) - f(x_0)$ and $f(x_0-h) - f(x_0)$ have the same sign, then $f'(x_0)$ should be zero, otherwise they will have different signs. Hence the necessary ~~and~~ condition that $f(x)$ should have local optimum value at any extreme point $x = x_0$ the first derivative $f'(x_0) = 0$.

Remark:- The distinction between a local minimum and local maximum can also be seen by examining the direction of change of first derivative $f'(x_0)$ at $x = x_0$

(i) If the sign of $f'(x_0)$ changes from positive to negative as x increases in the nbd of $x = x_0$, then the value of $f(x)$ will be a local maximum.

(ii) If the sign of $f'(x_0)$ changes from negative to positive as x increases in the nbd of $x = x_0$ then the value of $f(x)$ will be a local minimum.

Theorem 2:- (Sufficient Condition): If at an extreme point $x = x_0$ of $f(x)$, the first $(n-1)$ derivatives of it become zero i.e, $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then

(i) local max^m of $f(x)$ occurs at $x = x_0$ if $f^{(n)}(x_0) < 0$, for n even

(ii) local mini^m of $f(x)$ occurs at $x = x_0$ if $f^{(n)}(x_0) > 0$, for n even.

(iii) point of inflection occurs at $x = x_0$ if $f^{(n)}(x_0) \neq 0$, for n odd.

Q1. Find the second order Taylor's series approximation of the function

$f(x_1,x_2) = x_1^2 x_2 + 5x_1 e^{x_2}$ about the point $x_0 = [1,0]^T$

Soln:- The second order Taylor's series approximations of the function $f(x_1,x_2)$ at $x = x_0$ is

$$f(x_1,x_2) = f\begin{bmatrix}1\\0\end{bmatrix} + \nabla f\begin{bmatrix}1\\0\end{bmatrix} h + \frac{1}{2!} h^T H(\bar{x}) h$$

where $x = x_0 + \theta h$ and

$$h = \begin{bmatrix}h_1\\h_2\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x_1-1\\x_2\end{bmatrix}$$

$$x_0+\theta h = \begin{bmatrix}1\\0\end{bmatrix} + \theta\begin{bmatrix}x_1-1\\x_2\end{bmatrix} = \begin{bmatrix}1+\theta x_1-\theta\\\theta x_2\end{bmatrix}$$

$$\nabla f(x_0) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = \left(2x_1x_2 + 5e^{x_2},\ x_1^2 + 5x_1 e^{x_2}\right)$$

For $x_0 = [1,0]^T$, the value of $\nabla f(x_0) = [5,6]$.

$$H(\bar{x}) = \begin{bmatrix}\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2}\\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2}\end{bmatrix} = \begin{bmatrix}2x_2 & 2x_1+5e^{x_2}\\ 2x_1+5e^{x_2} & 5x_1 e^{x_2}\end{bmatrix}$$

Substituting the values in $f(x_1,x_2)$, we get

$$f(x_1,x_2) = 5 + [5,6]\begin{bmatrix}x_1-1\\x_2\end{bmatrix} + \frac{1}{\lfloor 2}\begin{bmatrix}x_1-1\\x_2\end{bmatrix}^T \begin{bmatrix}2x_2 & 2x_1+5e^{x_2}\\ 2x_1+5e^{x_2} & 5x_1 e^{x_2}\end{bmatrix}\begin{bmatrix}x_1-1\\x_2\end{bmatrix}$$

Q2. Examine the following functions for extreme points

(a) $f(x) = 4x^4 - x^2 + 5$ (b) $f(x) = (3x-2)^2(2x-3)^2$

(c) $f(x) = x^5/5 - 5x^4/2 + 35x^3/3 - 25x^2 + 24x$

(d) $f(x) = x^3 - 15x^2 + 10x + 100$.

(§) Constrained multivariate optimization with equality constraints :—

We shall discuss the problem of optimization of a continuous and differentiable objective function subject to equality constraints. That is

Optimize (max or min)

$$Z = f(x_1, x_2, \cdots, x_n)$$

subject to the constraints

$$g_i(x_1, x_2, \cdots, x_n) = 0; \quad i = 1, 2, \cdots, m$$

In matrix form we write,

optimize, $Z = f(\bar{x})$

subject to the constraints

$$g_i(\bar{x}) = 0, \; i = 1, 2, \cdots, m$$

where $\bar{x} = (x_1, x_2, \cdots, x_n)$, and $g_i(\bar{x}) = h_i(\bar{x}) - b_i$;

b_i is a constant

Here it is assumed that $m < n$ to get the solution.

There are various method of solution. But we will study only two. These are

(1) Substitution Method

(2) Lagrange multipliers method.

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(1) Direct Substitution Method :—

Since the constraints $g_i(\bar{x})$ is also continuous and differentiable, any variable in the constraint can be expressed in terms of the remaining variables. Then it is substituted in the objective function. The new objective function so obtained is not subject to any constraints and hence its optimum value can be obtained by the unconstrained optimization method as discussed in the previous class.

Some time this method is not convenient, particularly, when [redacted] there are more than two variables in the objective fns. and are subject to constraints.

Ex 1:- Find the optimum solution of the following constraint multivariable problem

Minimize $Z = x_1^2 + (x_2 + 1)^2 + (x_3 - 1)^2$

Subject to the constraint.

$$x_1 + 5x_2 - 3x_3 = 0$$

[redacted]

Soln :- Since the given problem has three variables and one equality constraints, any one of the variables can be removed from Z with the help of the equality constraints. Let us

[redacted]

Unconstrained Multivariate Optimization

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The possible geometries are:—

① If . Multi variate optimization means optimization of a scalar function of a several variables, — i.e. $y = P(\bar{x})$ and has the general form $\min_x P(\bar{x})$, where $P(\bar{x})$ is a nonlinear scalar-valued function of the vector variable $\bar{x}$

Background:— Before we discuss optimization methods, we need to talk about how to characterize nonlinear, multi variable functions such as $P(\bar{x})$. Let us consider the 2nd order Taylor series expansion about the point x_0:

$$P(\bar{x}) = P(\bar{x}_0) + \nabla_x P\Big|_{x_0}(\bar{x}-\bar{x}_0) + \frac{1}{2}(\bar{x}-\bar{x}_0)^T \nabla_x^2 P\Big|_{x_0}(\bar{x}-\bar{x}_0)$$

If we consider:—

$$a = P(\bar{x}_0) - \nabla_x P\Big|_{x_0} \bar{x}_0 + \frac{1}{2}(\bar{x}_0)^T \nabla_x^2 P\Big|_{x_0} \bar{x}_0$$

$$\bar{b}^T = \nabla_x P\Big|_{x_0} - (\bar{x}_0)^T \nabla_x^2 P\Big|_{x_0}$$

&

$$\vec{H} = \nabla_x^2 P\Big|_{x_0}$$

Then we can re-write the Taylor Series expansion as a quadratic approximation for $P(\bar{x})$:

$$P(\bar{x}) = a + b^T\bar{x} + \frac{1}{2}\bar{x}^T H \bar{x}$$

and the derivatives are

$$\nabla_x P(\bar{x}) \cong \bar{b}^T + x^T H \quad \leftarrow \text{gradient}$$

$$\nabla_x^2 P(\bar{x}) \cong H \quad \text{Hessian Matrix}$$

See for both

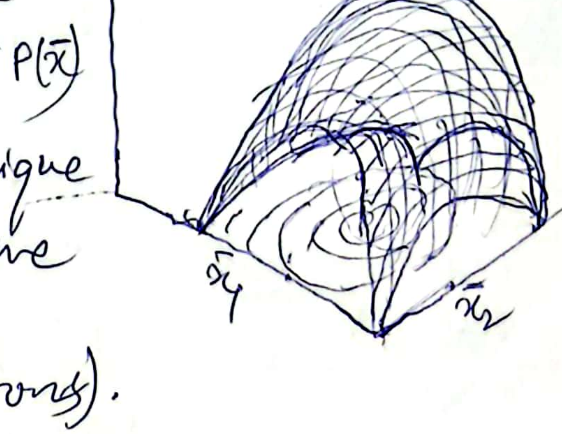
We can describe some of the local geometric (2) properties of $P(\bar{x})$ using its gradient and Hessian. In fact there are only a few possibilities for the local geometry, which can be easily be differentiated by the eigenvalues of the Hessian matrix $(\bar{H})$.

Recall that the eigenvalues of a square matrix $(\bar{H})$ are computed by finding all of the roots (λ_i) of the characteristic equation: $|\lambda \bar{I} - \bar{H}| = 0$ ← Mod

The possible geometries are:

① If $\lambda_i < 0$ $(\forall (i = 1, \ldots, n))$, the Hessian is said to be negative definite. This object has a unique maximum and is what we commonly refer to as a hill (in three dimensions). See fig-1.

② If $\lambda_i > 0$ $(\forall i = 1, \ldots, n)$, the Hessian matrix is said to be positive definite. This object has a unique minimum and is what we commonly refer to as a valley (in three dimensions).



A well posed problem has a unique optimum, so will limit our discussions to either problems with positive definite Hessian (for minimization) or negative definite Hessian (for maximization).

Further, we would prefer to choose units for our decision variable (x) so that the eigenvalues of the Hessian all have approximately the same magnitude. This will scale the problem so that the profit contours are concentric circles and will condition our optimization calculations.

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Q. Eigen values and Eigen vectors of matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Soln:— The characteristic eqn is

$|A - \lambda I| = 0$ i.e.,

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

or $(3-\lambda)(2-\lambda)(5-\lambda) = 0$

Thus the eigen values of A are 2, 3, 5.

If x, y, z are the components of an eigen vector corresponding to the eigen value λ, we have

$$|A - \lambda I| X = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Putting $\lambda = 2$ we have.

$x + y + 4z = 0$
$6z = 0$
$3z = 0$

$x + y = 0$ (1)
& $z = 0$.

from (1) $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0} = k_1$ (say).

Hence the eigen vector corresponding to $\lambda = 2$ is $k_1(1, -1, 0)$.

Putting $\lambda = 3$ we have

$y + 4z = 0$
$-y + 6z = 0$
$2z = 0$

$y = 0, z = 0$

$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} = k_2$

Hence the eigen vector corresponding to $\lambda = 3$ is $k_2(1, 0, 0)$

Illy the eigen vector corresponding to $\lambda = 5$ is $k_3(3, 2, 1)$.

(1) Any sq. matrix (A) & its transpose (A') have the same eigen values.

(2) The eigen values of a triangular matrix are just the diagonal elements of the matrix.

(3) The eigen values of an idempotent matrix are either zero or unity.

Idempotent mean $A^2 = A$.

(4) The sum of the eigen values of a matrix is the sum of elements of the principal diagonal.

or

$f_m(x_1, x_2, \ldots x_n)$ (≤) = 0 (%) Don ...

(6)

(vi) The product of the eigen values of a matrix A is equal to its determinant.

(vii) If λ is an eigen value of a matrix A, then $1/\lambda$ is the eigen value of A^{-1}

(viii) If λ is an eigen value of an orthogonal matrix, then $1/\lambda$ is also its eigen values.

A is an orthogonal iff $A^{-1} = A'$

(viii) If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ (m being a positive integer)

Thus eigenvalues of a square Hessian matrix (H) are computed by finding all of the roots (λ_i) of its characteristic eqn.

$$|\lambda I - H| = 0.$$

The possible geometries are:

(1) If $\lambda_i < 0$ ($\forall i = 1, \ldots n$), the Hessian is said to be negative definite. The object has a unique maximum and is what we commonly refer to as a hill (in three dimensions).

Example (1) Consider the scalar fn.

$$P(x) = 3 + x_1 + 2x_2 + 4x_1x_2 + x_1^2 + x_2^2$$

$$P(x) = 3 + [1 \ 2]\bar{x} + \bar{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \bar{x}$$

where $\bar{x} = [x_1, x_2]^T$

Stationarity of the gradient requires

$$\nabla_x P = [1 \ 2] + 2\bar{x}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = 0$$

$$x = -\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \end{bmatrix}$$

Check the $H(x)$ to classify the type of stationary point:-

$$\nabla_x^2 P = 2\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

The eigenvalues of $H(x)$ are

$$\left| \lambda I - \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \right| = \left| \begin{bmatrix} \lambda-2 & -4 \\ -4 & \lambda-2 \end{bmatrix} \right|$$

$$(\lambda-2)^2 - 16 = 0 \ \therefore \lambda_i = 6, -2$$

Thus the Hessian is indefinite and the st. pt. is a saddle point. This is a

General procedure for finding / determining the optimality conditions

(b) Constrained Multi-variable optimization with equality constraints :—

We consider the problem of optimizing a continuous and differentiable functions subject to equality constraints.

That is

Optimize (max or min) $z = f(x_1, x_2, \cdots, x_n)$

Subject to the constraints

$$g_i(x_1, x_2, \cdots, x_n) = 0, \quad i = 1, 2, \cdots, m$$

In matrix notation the above problem can also be written as:

Optimize (max or min) $z = f(\bar{x})$

Subject to the constraints $g_i(\bar{x}) = 0, \; i = 1, 2 \cdots m$

where, $\bar{x} = (x_1, x_2, \cdots, x_n)$,

and $g_i(\bar{x}) = h_i(\bar{x}) - b_i$;

Where, b_i is a constant. We assume that $m < n$ to get the solution.

There are various method for solving the above problems. There are :—

(a) Direct Substitution Method,

(b) Lagrange Multipliers Method.

(i) Direct Substitution Method :—

Since, the constraints set $g_i(\bar{x})$ is also continuous and differentiable, any variable in the constraint set can be expressed in terms of the remaining variables.

$g_m(x_1, x_2, \cdots x_n)$ $(\leq, =, \geq)$ b_m and

Then it is substituted in the objective function. The new objective function so obtained is not subject to any constraint and hence its optimum value can be obtained by the unconstrained optimization method discussed previously.

Necessary condition for a problem with $n=3$ and $m=1$ (13)

Optimize (max or min) $z = f(x_1, x_2, x_3)$, — (1)

Sub. to the constraints

$$g(x_1, x_2, x_3) = 0 \quad \text{— (2)}$$

Let an optimum value of z occur at a point $(x_1, x_2, x_3) = (a, b, c)$ at which atleast one of the partial derivatives

$\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}$ does not vanish. Thus we may proceed as follows:—

(i) We choose one variable say x_3 in constraint and express it in terms of the remaining two variables $\rightarrow x_3 = h(x_1, x_2)$

(ii) Then substitute the value of x_3 into the objective function (1) getting

$$z = f\{x_1, x_2, h(x_1, x_2)\}$$

From unconstrained optimization method, we know that the necessary condition for local optimum is that all first order derivatives must be zero

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Direct Substitution Method :-

Find the optimum solution of the following constrained multivariable problems.

Minimize $Z = x_1^2 + (x_2+1)^2 + (x_3-1)^2$

Subject to the constraints

$$x_1 + 5x_2 - 3x_3 = 6$$

Soln. Since the given problem has three variables and one equality constraints any one of the variables can be removed from the ob. fn Z with the help of the equality constraint. Let us choose variable x_3 to be eliminated from Z. Then from the equality constraints we have

$$x_3 = \frac{(x_1 + 5x_2 - 6)}{3}$$

Substituting the value of x_3 in the ob. fn we get

$$Z \text{ or } f(x) = x_1^2 + (x_2+1)^2 + \frac{1}{9}(x_1+5x_2-9)^2$$

The necessary condition for minimum of Z is that the gradient (unconstrained Multivariable optimization)

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right] = 0$$

That is
$$\frac{\partial Z}{\partial x_1} = 2x_1 + \frac{2}{9}(x_1+5x_2-9) = 0 \quad (1)$$

$$\frac{\partial Z}{\partial x_2} = 2(x_2+1) + \frac{10}{9}(x_1+5x_2-9) = 0 \quad (2)$$

ON Solving (1) & (2) we get $x_1 = 2/5$ & $x_2 = 1$

$g_m(x_1, x_2, \ldots x_n) \; (\leq, = 0 \ldots)$

To find whether the soln so obtained is minimum or ~~maximum~~ not, we apply the sufficiency condition by forming a Hessian matrix.

The Hessian matrix for the given objective fn is

$$H(x_1, x_2) = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 20/9 & 10/9 \\ 10/9 & 20/9 \end{bmatrix}$$

Since the matrix is symmetric and principal diagonal elements are positive, $H(x_1, x_2)$ is positive definite and the objective fn is convex (??). Hence the optimum solution to the given problem is

$x_1 = 4/5, x_2 = 1$.

$x_3 = -1/5$ and Min $Z = 28/5$

with respect to x_1 and x_2. that is 120 [14]

$$\frac{\partial z}{\partial x_j} = 0; \quad j = 1, 2 \quad (243) \text{ ——— } (37).$$

Applying the chain rule for differentiation on (37) we get

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_3} \cdot \frac{\partial h}{\partial x_j}; \quad j = 1, 2.$$

But from eqn (20), we have

$$\frac{\partial g}{\partial x_j} + \frac{\partial g}{\partial x_3} \cdot \frac{\partial h}{\partial x_j} = 0; \quad j = 1, 2$$

$$\frac{\partial h}{\partial x_j} = -\frac{(\partial g / \partial x_j)}{(\partial g / \partial x_3)}; \quad \frac{\partial g}{\partial x_3} \neq 0, \quad j = 1, 2$$

At the point $(x_1, x_2, x_3) = (a, b, c)$ ~~we have~~,

Since optimum occurs at the point (a, b, c) we have,

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial x_j} - \left[\frac{\partial f}{\partial x_3}\left\{\frac{\partial g / \partial x_j}{\partial g / \partial x_3}\right\}\right] = 0. \quad (?)$$

At $(x_1, x_2, x_3) = (a, b, c)$

As $\frac{\partial g}{\partial x_3} \neq 0$, we define a quantity λ, called Lagrange multiplier as given below. The value of λ represents the amount of change in the objective function due to per unit change in the constraint limit i.e,

$$\text{i.e.} \quad \frac{\partial f}{\partial x_3} - \lambda \frac{\partial g}{\partial x_j} = 0, \text{ at } (x_1, x_2, x_3) = (a, b, c)$$

in

$g_m (x_1, x_2, \ldots x_n) \ (\leq, =, \geq) \ b_m$ ...

$$x, \lambda = \frac{(\partial f/\partial x_3)}{(\partial g/\partial x_3)} \quad (2.4)$$

Equation (4) can now be written as

$$\frac{\partial z}{\partial x_j} = \left(\frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j}\right) = 0, \; j = 1, 2. \quad (5)$$

at $(x_1, x_2, x_3) = (a, b, c)$ and the constraint equation $g(x_1, x_2, x_3) = 0$. ——— (6)

is also satisfied at the extreme (or critical) points, $x_1 = a$, $x_2 = b$, $x_3 = c$. The condition (5) and (6) are called necessary conditions for a local optimum, provided not all $\left(\frac{\partial g}{\partial x_j}\right)$, $j = 1, 2$ become zero at the extreme points.

The necessary condition given by eqⁿ (5) & (6) can be obtained very easily by forming a function L called the Lagrangian function as

$$L(x_j, \lambda) = f(x_j) - \lambda g(x_j), \; j = 1, 2, 3. \quad (7)$$

Differentiating $L(x_j, \lambda)$ partially w.r.to x_j ($j = 1, 2, 3$) and λ and equating them to zero. Then the following equations provide the necessary conditions:—

$$\left.\begin{aligned} \frac{\partial L}{\partial x_j} &= \frac{\partial f}{\partial x_j} - \lambda \frac{\partial g}{\partial x_j} = 0, \; j = 1, 2, 3 \\ \frac{\partial L}{\partial \lambda} &= g(x_j) = 0, \; j = 1, 2, 3 \end{aligned}\right\} \quad (8)$$

for local optimum and can be solved for the unknown x_j ($j = 1, 2, 3$) and λ.

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Remark:— The necessary conditions so obtained become sufficient conditions for a maximum (or minimum) if $f(x)$ is concave (or convex) with equality constraints.

Example 1. Obtain necessary conditions for the optimum solution of the following problem

Minimize $f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$.

Subject to the constraint

$g(x_1, x_2) = x_1 + x_2 - 7 = 0$.

Soln:— Forming the Lagrangian function we obtain

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$$
$$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$$

The necessary conditions for the minimum of $f(x_1, x_2)$ are given by

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} &= 6e^{2x_1+1} - \lambda = 0 \quad \text{or} \quad \lambda = 6e^{2x_1+1} \\ \frac{\partial L}{\partial x_2} &= 2e^{x_2+5} - \lambda = 0 \quad \text{or} \quad \lambda = 2e^{x_2+5} \\ \frac{\partial L}{\partial \lambda} &= -(x_1 + x_2 - 7) = 0 \end{aligned}\right\}$$

On solving these three equations in three unknowns, we obtain

$$\left.\begin{aligned} x_1 &= \tfrac{1}{3}(11 - \log 3), \text{ and} \\ x_2 &= 7 - \tfrac{1}{3}(11 - \log 3) \end{aligned}\right\}$$

$g_m(x_1, x_2, \ldots x_n)\ (\leq, =, \geq)\ b_m$ ang

Remark:- The necessary conditions are obtained [illegible] becomes sufficient condition for a maximum (or minimum) of $f(x)$ [illegible] (or minimum) with equality constraints.

Example:- [illegible] necessary conditions for the extremum of the following problem.

Minimize $f(x_1, x_2) = 3e^{2x_1+1} + 2e^{x_2+5}$

subject to the constraint

$x_1 + x_2 = 7$

Solution:- [illegible] Lagrangian function we obtain

$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$

$= 3e^{2x_1+1} + 2e^{x_2+5} - \lambda(x_1 + x_2 - 7)$

The necessary conditions for the minimum of $f(x_1, x_2)$ are given by

$\frac{\partial L}{\partial x_1} = 6e^{2x_1+1} - \lambda = 0$ or $\lambda = 6e^{2x_1+1}$

$\frac{\partial L}{\partial x_2} = 2e^{x_2+5} - \lambda = 0$ or $\lambda = 2e^{x_2+5}$

$\frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 7) = 0$

Solving these equations in three unknowns we obtain

$x_1 = \frac{1}{3}(11 - \log 3)$ and

$[illegible] = \frac{1}{3}(11 - \log 3)$